Numer. Math. (2003)
Digital Object Identifier (DOI) 10.1007/s00211-003-0476-7

Numerische Mathematik

Almost optimal interior penalty discontinuous approximations of symmetric elliptic problems on non-matching grids

R.D. Lazarov[1], J.E. Pasciak[1], J. Schöberl[2], P.S. Vassilevski[3]

[1] Department of Mathematics, Texas A & M University, College Station, TX 77843, USA; e-mail: {lazarov,pasciak}@math.tamu.edu
[2] Institute für Mathematik, Johannes-Kepler-Universität, Altenberger Strasse 69, 4040 Linz, Austria; e-mail: joachim@sfb013.uni-linz.ac.at
[3] Center for Applied Scientific Computing, Lawrence Livermore National Laboratory, Mail Stop L-560, 7000 East Avenue, Livermore, CA 94550, USA; e-mail: panayot@llnl.gov

Received September 21, 2001 / Revised version received January 21, 2003 / Published online: August 5, 2003 –

Summary. We consider an interior penalty discontinuous approximation for symmetric elliptic problems of second order on non-matching grids in this paper. The main result is an almost optimal error estimate for the interior penalty approximation of the original problem based on partitioning of the domain into a finite number of subdomains. Further, an error analysis for the finite element approximation of the penalty formulation is given. Finally, numerical experiments on a series of model second order problems are presented.

Mathematics Subject Classification (2000): 65F10, 65N20, 65N30

1 Introduction

In this paper, we propose and analyze a simple strategy for constructing composite discretizations of self-adjoint second order elliptic equations on non-matching grids. The need for discretizations on non-matching grids is

Correspondence to: R.D. Lazarov

The work of the first and the second authors has been partially supported by the National Science Foundation under Grant DMS-9973328. The work of the last author was performed under the auspices of the U. S. Department of Energy by University of California Lawrence Livermore National Laboratory under contract W-7405-Eng-48.

motivated partially from the desire for parallel discretization methods (including adaptive) for PDEs, which is a much easier task if non-matching grids are allowed across the subdomain boundaries. Another situation may arise when different discretization techniques are utilized in different parts of the domain.

In the present paper, we consider a model situation when the domain is split into a fixed number of non-overlapping subdomains and each subdomain is meshed independently. This is a non-conforming method and the functions are, in general, discontinuous across the subdomain interfaces. The jump in the values of the function along the interface is "penalized" in the variational formulation, a standard approach in the interior penalty method (cf. [2], [4], [15], [25]).

Formulations that impose various constraints by penalty terms have been used in engineering computations and mathematically justified in the pioneering works of Lions [22], Babuška [5], and Nitsche [24]. For a recent comprehensive survey on this subject, see [3]. An important feature of the method we consider is that the term in the weak formulation involving the co-normal derivative of the solution on the interface boundaries is omitted. Such terms are added to the functional for consistency but often lead to a non-symmetric discretization (cf. [25]) of the original symmetric positive definite problem.

An alternative technique for dealing with non-matching grids involves the use of Lagrange multipliers or mortar spaces. There are a vast number of publications devoted to the mortar finite element method as a general strategy for deriving discretization methods on non-matching grids. We refer the interested reader to the series of Proceedings of the International Conferences on Domain Decomposition Methods (cf. [6], [12], [18] [for more information see, http://www.ddm.org]).

The motivation for studying this method, even though its convergence order is limited, is that it has some advantages over the competing methods. For example, mortar discretizations and discontinuous Galerkin methods lead to linear systems that are more difficult to solve. The method discussed here leads to a symmetric algebraic problem with optimal conditioning.

The method we consider, the interior penalty finite element approximation, was studied and tested on various examples in [21]. The error estimates derived in [21] were suboptimal with a loss of a factor $h^{1/2-\delta},\ \ 0 < \delta < 1/2$ for solutions in the Sobolev space $H^{2-\delta}(\Omega)$. In this paper we present a refined analysis and get almost optimal error estimates for linear finite element and solutions in $H^{2-\delta}(\Omega)$. In addition, we extend the analysis to decompositions with cross points.

In the case of matching grids, the finite element Galerkin method with penalty for a class of problems with discontinuous coefficients (interface problem) has been studied in [4]. Similarly, in [11], the interface problem has been addressed by recasting the problem as a first order system (by introducing the gradient of the solution as a new vector variable) and applying the least-squares method to the system. Integrals of the squared jump in the scalar and the normal component of the vector functions on the interface are added as penalty terms in the least-squares functional. In both cases, an optimal order method leads to discrete problem with non-optimal condition number.

Other approaches for handling discretizations on non-matching grids involve different discretizations in the different subdomains, for example, a mixed finite element method in one subdomain coupled with a standard Galerkin method in the other (proposed in [28] and studied further in [19]), a mixed finite element method coupled with a discontinuous Galerkin method (cf. [14]) or mixed finite element discretizations in both subdomains (cf. [1], [20]). Similarly, the coupling finite volume method and the Galerkin methods was proposed and studied in [16].

The structure of the present paper is as follows. In Section 2, we formulate the problem. In Section 3, we introduce the primal and dual penalty formulations of the problem split into subproblems on non-overlapping subdomains. To get an optimal error estimate, we introduce the mixed formulation of the penalty problem and derive a fundamental *a priori* error estimate for its solution (in Section 4). In Section 5, we analyze the difference between the solution of the original problem and the solution of the penalty formulation. The error is shown to be of almost optimal order for $u \in H^{2-\delta}(\Omega)$ for $\delta \geq 0$. For methods without cross-points, the error is optimal for $1/2 > \delta > 0$. Finally, the finite element discretization and its error analysis is presented in Section 6. Numerical tests illustrating the accuracy of the method for two model problems are given there as well.

2 Notations and problem formulation

In this paper we use the standard notation for Sobolev spaces of functions defined in a bounded domain $\Omega \subset \mathcal{R}^d$, $d = 2, 3$. For example, $H^s(\Omega)$ for s integer denotes the Hilbert space of functions u defined on Ω and having generalized derivatives up to order s that are square integrable in Ω. For non-integer $s > 0$, the spaces are obtained by the real method of interpolation (cf. [23]). $H^1_0(\Omega)$ is the space of functions in $H^1(\Omega)$, which vanish on $\partial\Omega$. The norm of $u \in H^s(\Omega)$ is denoted by $||u||_{s,\Omega}$. We also use the notation $|u|_{s,\Omega}$ for the s-order semi-norm. For the traces of functions in $H^1_0(\Omega)$ on a manifold Γ of dimension $d-1$ (curves and surfaces) and $\partial\Gamma \subset \partial\Omega$, we will sometimes use the fractional order Sobolev spaces commonly denoted by $H^{1/2}_{00}(\Gamma)$, which is defined to be the interpolation space halfway between $H^1_0(\Gamma)$ and $L^2(\Gamma)$.

For a given Hilbert space H with an inner product $(\cdot, \cdot)_H$ and corresponding norm $\|\cdot\|_H$, we denote H^* to be the dual, i.e., the space of all continuous linear functionals on H. We use the fact that $(H_{00}^{1/2}(\Gamma))^* = H^{-1/2}(\Gamma)$.

For a given bounded polygon (polytope) Ω, a source term $f \in L^2(\Omega)$ and a symmetric and uniformly positive definite and bounded coefficient matrix $a(x)$ on Ω, we consider the following model boundary value problem in a weak form: find $u \in H_0^1(\Omega)$ such that:

$$A(u, v) = f(v) \quad \text{for all } v \in H_0^1(\Omega). \tag{2.1}$$

Here $A(u, v) = \int_\Omega a\nabla u \cdot \nabla v \, dx$ and $f(v) = (f, v)_{0,\Omega} := \int_\Omega f v \, dx$.

3 Interior penalty formulation

We shall study a discretization of this problem by the finite element method using meshes that generally do not align along certain interfaces. This situation arises when the domain Ω is split into p non-overlapping subdomains $\Omega_i, \quad i = 1, \dots, p$, and each subdomain is meshed (triangulated) independently of the others. We assume that the number of subdomains is fixed and each subdomain is a shape regular polyhedron. A model situation of this type for $d = 2$ is shown in Figure 1. We denote by γ_{ij} the interface between two subdomains Ω_i and Ω_j and by Γ the union of all interfaces γ_{ij}.

We define

$$\begin{aligned}
V &:= \{v \in L_2(\Omega) : v|_{\Omega_i} \in H^1(\Omega_i) \cap H_0^1(\Omega)\},\\
Q &:= L^2(\cup \partial\Omega_i),\\
a(u, v) &:= \sum_i (a\nabla u, \nabla v)_{0,\Omega_i} := \sum_i a_i(u, v),\\
c(p, q) &:= (p, q)_{0,\Gamma} := \int_\Gamma pq \, ds,\\
(p, q)_{0,\gamma_{ij}} &:= \int_{\gamma_{ij}} pq \, ds, \text{ and}\\
\Lambda\, u &:= [u]_\Gamma.
\end{aligned}$$

Here the jump $[u]$ is defined as the difference of the traces of a function u on Γ. We specify a "master" side of each interface γ_{ij} so on γ_{ij} the jump is defined always as $[u]_{\gamma_{ij}} = u|_{\Omega_i} - u|_{\Omega_j}$, where Ω_i is the domain from the master side of γ_{ij}.

We approximate the original problem (2.1) by the following problem (which we subsequently call the interior penalty formulation): Find $u_\epsilon \in V$ such that

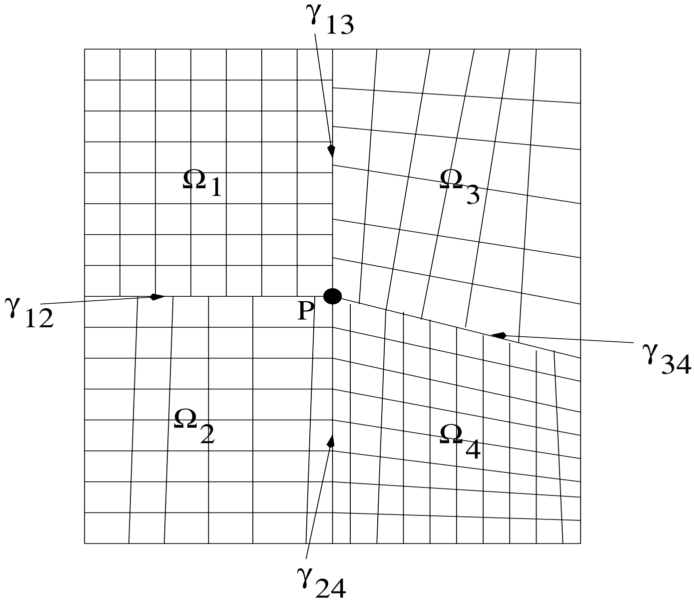

Fig. 1. A domain Ω partitioned into four subdomains Ω_i $i = 1, 2, 3, 4$ with interfaces γ_{ij}; each subdomain is partitioned into quadrilateral finite elements independently; P is a cross point.

$$A^{\epsilon}(u_\epsilon, \varphi) := a(u_\epsilon, \varphi) + \epsilon^{-1} c(\Lambda u_\epsilon, \Lambda\varphi) = f(\varphi), \qquad \text{for all } \varphi \in V. \tag{3.1}$$

Here ϵ is a small parameter that later will be chosen as the mesh size of the finite element partition of Ω. The problem (3.1) is also called the primal formulation to distinguish it from the the mixed formulation introduced in the next section.

The formulation (3.1) allows discontinuous solutions along the interface Γ. We have introduced a penalty term with a large parameter ϵ^{-1} to control the size of the jump $[u_\epsilon]_\Gamma$. Our goal is to estimate the difference $u - u_\epsilon$ assuming that $u \in H^{2-\delta}(\Omega)$ for $0 \le \delta < 1/2$.

The bilinear form $A^{\epsilon}(\cdot, \cdot)$ defined in (3.1) is symmetric and positive definite. It is related to, but much simpler than, the corresponding discontinuous Galerkin method used in [2], [25]. The simplification comes from the fact that we do not have a term involving the co-normal derivative $a\nabla u \cdot \mathbf{n}$ along the interface Γ (here $\mathbf{n}$ is the unit normal vector along Γ). This simplification comes at a cost: the proposed approximation will have almost optimal order of convergence for linear elements only. In contrast,the non-symmetric interior penalty Galerkin method studied in [25] has optimal order estimates for continuous finite elements of any degree. However, our formulation leads to a symmetric and positive definite problem, which is more convenient for computational purposes.

4 Study of the primal and mixed formulations

In this section, we shall study the solution of (3.1). This problem fits into a general abstract class of parameter dependent problems which we now describe. Let $(V, \|\cdot\|_V, (\cdot,\cdot)_V)$ and $(Q, \|\cdot\|_c, c(\cdot,\cdot))$ be Hilbert spaces with their respective norms and inner products. We assume that we are given a continuous symmetric positive semi-definite bilinear form $a(\cdot,\cdot)$ on $V \times V$ and a continuous linear map $\Lambda : V \to Q$ so that

$$a(v, w) \preceq ||v||_V ||w||_V \quad \text{and} \quad \|\Lambda v\|_c \preceq ||v||_V, \text{ for all } v, w \in V.$$

Here, and in the rest of the paper, we use the signs $\preceq$ and $\succeq$ to denote inequalities with a constant that might depend on various parameters but is independent of ϵ.

Next, we define

$$A^\epsilon(v, w) = a(v, w) + \epsilon^{-1} c(\Lambda v, \Lambda w), \text{ for all } v, w \in V.$$

We assume that the range of Λ is dense in Q but not necessarily closed. The parameter $\epsilon \in (0, 1]$ is typically small. We further assume that $A^1(\cdot,\cdot)$ gives rise to an equivalent norm on V, i.e.,

$$\|v\|_V^2 \preceq A^1(v, v) \preceq \|v\|_V^2, \text{ for all } v \in V. \tag{4.1}$$

It easily follows that $A^\epsilon(\cdot,\cdot)$ is coercive on V and satisfies

$$\|v\|_V^2 \preceq A^\epsilon(v, v) \preceq \epsilon^{-1}\|v\|_V^2, \text{ for all } v \in V.$$

Our approach is to reformulate (3.1) as a mixed problem following [27]. Let u_ϵ be the solution of (3.1) and define the dual variable $p_\epsilon \in Q$ by

$$p_\epsilon := \epsilon^{-1}\Lambda u_\epsilon. \tag{4.2}$$

We get the mixed system for u_ϵ and p_ϵ:

$$a(u_\epsilon, v) + c(\Lambda v, p_\epsilon) = f(v) \quad \text{for all } v \in V, \tag{4.3}$$

$$c(\Lambda u_\epsilon, q) - \epsilon\, c(p_\epsilon, q) = 0 \quad \text{for all } q \in Q. \tag{4.4}$$

Combining equations (4.3) and (4.4), and introducing the product space $X \equiv V \times Q$, we obtain the mixed variational problem: Find $(u_\epsilon, p_\epsilon) \in X$ satisfying

$$B^\epsilon((u_\epsilon, p_\epsilon), (v, q)) = f(v) \qquad \text{for all } (v, q) \in X, \tag{4.5}$$

with the block bilinear form

$$B^\epsilon((u_\epsilon, p_\epsilon), (v, q)) := a(u_\epsilon, v) + c(\Lambda u_\epsilon, q) + c(\Lambda v, p_\epsilon) - \epsilon\, c(p_\epsilon, q). \tag{4.6}$$

The mixed bilinear form is well defined for the limit $\epsilon = 0$.

Any solution (u_ϵ, p_ϵ) of (4.5) is in the space

$$X_0 = \{(v, q) \in X : \Lambda v = \epsilon\, q\}\,. \tag{4.7}$$

This space will play an essential role in the analysis of the proposed interior penalty method. On the space X, we define the norm

$$\|(u, p)\|_\epsilon := (\|u\|_V^2 + \epsilon\|p\|_c^2)^{1/2}. \tag{4.8}$$

This norm degenerates to a semi-norm for $\epsilon = 0$. The bilinear form $B^\epsilon(\cdot, \cdot)$ is continuous with parameter dependent bounds for that norm, namely, for $(u, p) \in X$ and $(v, q) \in X$

$$\begin{aligned} B^\epsilon((u, p), (v, q)) &= a(u, v) + c(\Lambda u, q) + c(\Lambda v, p) - \epsilon\, c(p, q) \\ &\preceq \left(a(u, u) + \|\Lambda u\|_c^2 + \|p\|_c^2 + \epsilon\, \|p\|_c^2\right)^{1/2} \\ &\quad \times \left(a(v, v) + \|\Lambda v\|_c^2 + \|q\|_c^2 + \epsilon\, \|q\|_c^2\right)^{1/2} \\ &\preceq \epsilon^{-1}\, \|(u, p)\|_\epsilon\, \|(v, q)\|_\epsilon . \end{aligned} \tag{4.9}$$

On the other hand, $B^\epsilon(\cdot, \cdot)$ provides a uniformly continuous mapping from the dual of X (with respect to the norm $\|(\cdot, \cdot)\|_\epsilon$) into X. This is formulated in the following theorem:

Theorem 4.1 *Let f and g be continuous linear functionals on V and Q, respectively. Then the extended mixed problem:*

$$B^\epsilon((u, p), (v, q)) = f(v) + g(q) \qquad \text{for all } (v, q) \in V \times Q \tag{4.10}$$

has a unique solution $(u, p) \in X$. Moreover,

$$\|u\|_V^2 + \epsilon^{-1}\|\Lambda u\|_c^2 + \epsilon\|p\|_c^2 \preceq \|f\|_{V^*}^2 + \epsilon^{-1}\|g\|_{Q^*}^2. \tag{4.11}$$

Here $\|f\|_{V^*}$ and $\|g\|_{Q^*}$ denote the norms of the linear functionals.

Proof. First, we construct a solution by means of the primal problem. Since $\Lambda : V \to Q$ is continuous, and $g(\cdot)$ is in Q^*, the functional $g(\Lambda\cdot)$ is continuous on V:

$$|g(\Lambda v)| \le \|g\|_{Q^*}\|\Lambda v\|_c \preceq \|g\|_{Q^*}\|v\|_V.$$

Let $u \in V$ be the solution of

$$a(u, v) + \epsilon^{-1}c(\Lambda u, \Lambda v) = f(v) + \epsilon^{-1}g(\Lambda v) \quad \text{ for all } v \in V.$$

We use (4.1) to get

$$\begin{aligned} \|u\|_V^2 + \epsilon^{-1}\|\Lambda u\|_c^2 &\preceq a(u, u) + \epsilon^{-1}\, c(\Lambda u, \Lambda u) \\ &= f(u) + \epsilon^{-1}g(\Lambda u) \\ &\preceq \|f\|_{V^*}\|u\|_V + \epsilon^{-1/2}\|g\|_{Q^*}\, \epsilon^{-1/2}\|\Lambda u\|_c \\ &\preceq \left(\|f\|_{V^*}^2 + \epsilon^{-1}\|g\|_{Q^*}^2\right)^{1/2} \left(\|u\|_V^2 + \epsilon^{-1}\|\Lambda u\|_c^2\right)^{1/2} . \end{aligned}$$

Dividing by $(\|u\|_V^2 + \epsilon^{-1}\|\Lambda u\|_c^2)^{1/2}$ gives the bound for u. By the Riesz Representation Theorem, we define $\tilde{g} \in Q$ such that

$$c(\tilde{g}, q) = g(q) \quad \text{for all } q \in Q$$

and find that

$$p = \epsilon^{-1}(\Lambda u - \tilde{g}).$$

Clearly,

$$\epsilon \|p\|_c^2 \preceq \epsilon^{-1}\|\Lambda u\|_c^2 + \epsilon^{-1}\|g\|_{Q^*}^2 \preceq \|f\|_{V^*}^2 + \epsilon^{-1}\|g\|_{Q^*}^2.$$

We verify that (u, p) is a solution of (4.10). Indeed, for all $(v, q) \in X$,

$$\begin{aligned} B^\epsilon((u, p), (v, q)) &= a(u, v) + c(\Lambda u, q) + c(\Lambda v, \epsilon^{-1}(\Lambda u - \tilde{g})) \\ &\quad -\epsilon c(\epsilon^{-1}(\Lambda u - \tilde{g}), q) \\ &= a(u, v) + \epsilon^{-1} c(\Lambda u, \Lambda v) - \epsilon^{-1} c(\tilde{g}, \Lambda v) + c(\tilde{g}, q) \\ &= f(v) + g(q). \end{aligned}$$

Finally, we prove that the solution is unique. Any solution (u, p) of the homogeneous problem satisfies

$$\begin{aligned} 0 &= B^\epsilon((u, p), (u, \Lambda u - p)) \\ &= a(u, u) + c(\Lambda u, \Lambda u) + \epsilon\, c(p, p) - \epsilon\, c(p, \Lambda u) \\ &\geq a(u, u) + (1 - \frac{\epsilon}{2})\, c(\Lambda u, \Lambda u) + \frac{\epsilon}{2}\, c(p, p). \end{aligned}$$

Thus, zero is the only solution of the homogeneous equation and the proof is complete. □

We will now demonstrate the benefit in using the mixed form. Namely, in Theorem 4.3 we will show an *a priori* estimate of the solution to the problem (4.10) that is uniform in $\epsilon > 0$.

Define the norm $\|p\|_{Q,0}$ for $p \in Q$ by

$$\|p\|_{Q,0} = \sup_{v \in V} \frac{c(p, \Lambda v)}{\|v\|_V}. \tag{4.12}$$

This is a norm since ΛV is dense in Q. Further, denote by Q_0 the closure of ΛV in the norm $\|\cdot\|_{Q,0}$. In general, $\|.\|_{Q,0}$ is a weaker norm than $\|.\|_c$. By definition, Λ has a closed range in Q_0^*. In the limit case ($\epsilon = 0$) the bilinear form $B^\epsilon((u, p), (v, q))$ is continuous and stable on $V \times Q_0$:

Theorem 4.2 (Brezzi, e.g., [10], Proposition 1.3) *The bilinear form*

$$B^0((u,p),(v,q)) = a(u,v) + c(\Lambda u, q) + c(\Lambda v, p)$$

is continuous, i.e.,

(4.13) $$B^0((u,p),(v,q)) \preceq (\|u\|_V^2 + \|p\|_{Q,0}^2)^{1/2}(\|v\|_V^2 + \|q\|_{Q,0}^2)^{1/2},$$

and stable, i.e.,

(4.14) $$\sup_{u\in V,\ p\in Q_0} \frac{B^0((u,p),(v,q))}{(\|u\|_V^2 + \|p\|_{Q,0}^2)^{1/2}} \succeq (\|v\|_V^2 + \|q\|_{Q,0}^2)^{1/2},$$

on the space $V \times Q_0$.

For the case $\epsilon > 0$, we need a norm that is ϵ–dependent. Namely, we define

(4.15) $$\|p\|_Q := \|p\|_{Q,\epsilon} := \left(\|p\|_{Q,0}^2 + \epsilon\|p\|_c^2\right)^{1/2}.$$

This norm is equivalent to $\|.\|_c$ for fixed $\epsilon > 0$, but not necessarily uniformly with respect to ϵ since obviously $\epsilon\|p\|_c^2 \le \|p\|_Q^2$. We define the product space

$$\mathcal{X} = V \times Q$$

with the norm

(4.16) $$\|(u,p)\|_{\mathcal{X}} = (\|u\|_V^2 + \|p\|_Q^2)^{1/2}.$$

The following theorem states that $B^\epsilon(\cdot,\cdot)$ is bounded in $\mathcal{X}$ and satisfies an *inf-sup* condition with a constant independent of ϵ:

Theorem 4.3 *Assume that (4.1) is satisfied. Let $B^\epsilon(\cdot,\cdot)$ and $\|\cdot\|_{\mathcal{X}}$ be defined by (4.5) and (4.16), respectively. Then:*

- *The bilinear form $B^\epsilon(\cdot,\cdot)$ is uniformly continuous on $\mathcal{X}$, i.e.,*

(4.17)
$$B^\epsilon((u,p),(v,q)) \preceq \|(u,p)\|_{\mathcal{X}}\, \|(v,q)\|_{\mathcal{X}} \quad \textit{for all } (u,p),\ (v,q) \in \mathcal{X};$$

- *The bilinear form $B^\epsilon(\cdot,\cdot)$ is uniformly stable on $\mathcal{X}$, i.e.,*

(4.18) $$\sup_{(u,p)\in\mathcal{X}} \frac{B^\epsilon((u,p),(v,q))}{\|(u,p)\|_{\mathcal{X}}} \succeq \|(v,q)\|_{\mathcal{X}} \qquad \textit{for all } (v,q) \in \mathcal{X};$$

- *the mixed problem $B^\epsilon((u,p),(v,q)) = f(v) + g(q)$ for all $(v,q) \in V \times Q$ has unique solution for any $f \in V^*$ and $g \in Q^*$ and the solution satisfies the a priori estimate:*

(4.19) $$\|(u,p)\|_{\mathcal{X}} \preceq \|f\|_{V^*} + \|g\|_{Q^*}.$$

Proof. The proof of the continuity follows the steps of the proof of estimate (4.9) but because of the new stronger norm in Q (see (4.15)), we have an improved estimate for the mixed term:

$$c(\Lambda u, q) \le \|u\|_V \sup_{v\in V} \frac{c(\Lambda v, q)}{\|v\|_V} = \|u\|_V \, \|q\|_{Q,0}$$

Thus, we get uniform continuity.

We need only to verify (4.18). To this end, fix $(v, q) \in \mathcal{X}$. By definition of the norm $\|.\|_{Q,0}$, there exists a $\tilde{v} \in V$ such that

$$\frac{c(\Lambda\tilde{v}, q)}{\|\tilde{v}\|_V} \succeq \|q\|_{Q,0}.$$

We are free to scale $\tilde{v}$ in such a way that

$$\|\tilde{v}\|_V = \|q\|_{Q,0} \qquad \text{and} \qquad c(\Lambda\tilde{v}, q) \succeq \|q\|^2_{Q,0}.$$

Let $(\tilde{u}, \tilde{p})$ be the unique solution (by Theorem 4.1) of

(4.20)
$$B^\epsilon((\tilde{u}, \tilde{p}), (w, r)) = (v, w)_V + c(\Lambda\tilde{v}, r) + \epsilon\, c(q, r) \quad \text{for all } (w, r) \in \mathcal{X}.$$

We will use $(\tilde{u}, \tilde{p})$ to verify (4.18). First, we see that

$$\begin{aligned} B^\epsilon((\tilde{u}, \tilde{p}), (v, q)) &= (v, v)_V + c(\Lambda\tilde{v}, q) + \epsilon\, c(q, q) \\ &\succeq \|v\|^2_V + \|q\|^2_{Q,0} + \epsilon\, \|q\|^2_c \\ &= \|(v, q)\|^2_{\mathcal{X}}, \end{aligned} \tag{4.21}$$

so that

$$\sup_{(u,p)\in\mathcal{X}} \frac{B^\epsilon((u, p), (v, q))}{\|(u, p)\|_{\mathcal{X}}} \ge \frac{B^\epsilon((\tilde{u}, \tilde{p}), (v, q))}{\|(\tilde{u}, \tilde{p})\|_{\mathcal{X}}} \succeq \frac{\|(v, q)\|^2_{\mathcal{X}}}{\|(\tilde{u}, \tilde{p})\|_{\mathcal{X}}}.$$

Thus, we need only to show that

$$\|(\tilde{u}, \tilde{p})\|_{\mathcal{X}} \preceq \|(v, q)\|_{\mathcal{X}}.$$

By the definition of $B^\epsilon(\cdot, \cdot)$ and (4.20), for all $(w, r) \in \mathcal{X}$,

$$\begin{aligned} B^\epsilon((\tilde{u} - \tilde{v}, \tilde{p}), (w, r)) &= B^\epsilon((\tilde{u}, \tilde{p}), (w, r)) - B^\epsilon((\tilde{v}, 0), (w, r)) \\ &= (v, w)_V + c(\Lambda\tilde{v}, r) + \epsilon\, c(q, r) \\ &\quad - [a(\tilde{v}, w) + c(\Lambda\tilde{v}, r)] \\ &= (v, w)_V - a(\tilde{v}, w) + \epsilon\, c(q, r). \end{aligned}$$

Applying Theorem 4.1 gives

$$\|\tilde{u} - \tilde{v}\|^2_V + \epsilon\, \|\tilde{p}\|^2_c \preceq \|v\|^2_V + \|\tilde{v}\|^2_V + \epsilon \|q\|^2_c.$$

Thus,

$$\|\tilde{u}\|_V^2 + \epsilon \|\tilde{p}\|_c^2 \preceq \|v\|_V^2 + \|q\|_{Q,0}^2 + \epsilon \|q\|_c^2 = \|(v, q)\|_{\mathcal{X}}^2.$$

Finally, we need to estimate $\|\tilde{p}\|_{Q,0}$. Using $(w, 0)$ in (4.20) gives

$$B^\epsilon((\tilde{u}, \tilde{p}), (w, 0)) \equiv a(\tilde{u}, w) + c(\Lambda w, \tilde{p}) = (w, v)_V, \text{ for all } w \in V.$$

Consequently,

$$\|\tilde{p}\|_{Q,0} = \sup_{w\in V} \frac{c(\Lambda w, \tilde{p})}{\|w\|_V} = \sup_{w\in V} \frac{(w, v)_V - a(\tilde{u}, w)}{\|w\|_V}$$

$$\preceq \|v\|_V + \|\tilde{u}\|_V \preceq \|(v, q)\|_{\mathcal{X}}.$$

Combining the above estimates completes the proof. □

5 Analysis of the interior penalty approximation

In this section, we derive the basic error estimates for the proposed interior penalty method (3.1). We present the estimate for the general case when the partition of Ω into subdomains Ω_i has "cross-points" (see Figure 1). For $d = 2$, the cross-points are the end points of the edges γ_{ij} that are in the interior of Ω. For $d = 3$, the cross-points are the edges of γ_{ij} that are in the interior of Ω. The analysis of the case without cross-points is somewhat simpler and is discussed at the end of this section.

We use some fundamental results from the domain decomposition literature (see [7,8]). Since all subdomains Ω_i are shape regular, the estimate

$$\|v|_{\gamma_{ij}}\|_{H_{00}^{1/2}(\gamma_{ij})} \preceq \|v\|_{H^1(\Omega_i)} \tag{5.1}$$

holds for functions $v \in H^1(\Omega_i)$ which vanish on $\partial\Omega_i \setminus \gamma_{ij}$. Here, $v|_{\gamma_{ij}}$ is the trace of v on γ_{ij}. We note also that given any $\sigma_{ij} \in H_{00}^{1/2}(\gamma_{ij})$, there is an extension v satisfying the above estimates. The following proposition plays a key role in the proof of the error estimate for the interior penalty method.

Proposition 5.1 *For any $\epsilon > 0$ and $\lambda \in L^2(\Gamma)$ with $\lambda|_{\gamma_{ij}} \in H^{1/2}(\gamma_{ij})$, the following estimate is valid:*

$$\|\lambda\|_{Q^*} \leq C \log \epsilon^{-1} \Big(\sum_{\gamma_{ij}} \|\lambda\|_{H^{1/2}(\gamma_{ij})}^2 \Big)^{1/2}. \tag{5.2}$$

The constant C is independent of ϵ but depends on the shape and the number of subdomains.

The proof of this estimate is given at the end of this section. Using it, we now prove the main result in this section:

Theorem 5.1 *Assume that the solution u of (2.1) is $H^{2-\delta}(\Omega)$–regular for some $\delta \in [0, 1/2)$. Then*

$$(5.3) \qquad ||u - u_\epsilon||_V \le C\epsilon^{1-\delta}(\log \epsilon^{-1})^{1-2\delta}||u||_{H^{2-\delta}(\Omega)}, \quad 0 \le \delta \le 1/2.$$

Here, the constant C is independent of ϵ.

Proof. We first note that the solution u of the problem (2.1) satisfies the identity,

$$(5.4) \qquad A^\epsilon(u,\ \varphi) = f(\varphi) + c(a\nabla u \cdot \mathbf{n}, \Lambda\varphi), \qquad \text{for all } \varphi \in V,$$

where the normal vector $\mathbf{n}$ is always pointing outward from the master side of γ_{ij}. Here, we have used the fact that the exact solution has continuous normal flux, i.e., in particular, $[a\nabla u \cdot \mathbf{n}]|_\Gamma = 0$. To simplify the notations, we define the function

$$\theta = a\nabla u \cdot \mathbf{n} \text{ on } \Gamma.$$

Subtracting (3.1) and (5.4) gives the following equation for the error $e = u - u_\epsilon$:

$$A^\epsilon(e,\ \varphi) = c(\theta, \Lambda\varphi) \quad \text{for all } \varphi \in V.$$

To use the *a priori* estimates of the mixed setting, we shall put this problem again in a mixed form. Namely, we introduce a new dependent variable $E := a\nabla u \cdot \mathbf{n} - \epsilon^{-1}\Lambda e = \theta - \epsilon^{-1}\Lambda e$ defined on Γ so that the pair (e, E) satisfies:

$$B^\epsilon((e, E), (v, q)) = \epsilon c(\theta, q) \qquad \text{for all } (v, q) \in V \times Q.$$

The estimate (4.19) will provide a basis for the analysis of the error (e, E), namely,

$$(5.5) \qquad ||e||_V + ||E||_Q \preceq \epsilon \sup_{q\in Q} \frac{c(\theta, q)}{||q||_Q}.$$

Because $||q||_Q \ge \epsilon^{1/2}||q||_c$, we easily get

$$(5.6) \qquad ||e||_V + ||E||_Q \preceq \sqrt{\epsilon}||\theta||_{0,\Gamma}.$$

The above estimate is an easy corollary of the set up of the problem but it yields an error for the interior penalty method of order at most $O(\epsilon^{1/2})$. We can improve it when θ is a smoother function. To accomplish this, we first apply estimate (5.2) for $\lambda = \theta$ to get:

$$(5.7) \qquad ||e||_V + ||E||_Q \preceq \epsilon\|\theta\|_{Q^*} \preceq \epsilon \log \epsilon^{-1}\Big(\sum_{\gamma_{ij}} \|\theta\|^2_{H^{1/2}(\gamma_{ij})}\Big)^{1/2}.$$

Second, we interpolate between the spaces $L^2(\gamma_{ij})$ and $H^{1/2}(\gamma_{ij})$ to get

$$\|\theta\|_{H^{1/2-\delta}(\gamma_{ij})} \preceq \|\theta\|^{1-2\delta}_{H^{1/2}(\gamma_{ij})} \|\theta\|^{2\delta}_{0,\gamma_{ij}}.$$

Next, we observe that interpolated norm with $\delta \in [0, 1/2]$ between

$$\left(\sum_{\gamma_{ij}} \|\theta\|^2_{0,\gamma_{ij}}\right)^{1/2} \quad \text{and} \quad \left(\sum_{\gamma_{ij}} \|\theta\|^2_{H^{1/2}(\gamma_{ij})}\right)^{1/2}$$

$$\text{is bounded by} \left(\sum_{\gamma_{ij}} \|\theta\|^2_{H^{1/2-\delta}(\gamma_{ij})}\right)^{1/2}.$$

This fact follows from the definition of the real interpolation method [23].

Finally, for $u \in H^{2-\delta}(\Omega)$, $0 \le \delta \le 1/2$, one can show that

$$\|\theta\|_{H^{1/2-\delta}(\gamma_{ij})} = \|a\nabla u \cdot \mathbf{n}\|_{H^{1/2-\delta}(\gamma_{ij})} \preceq \|u\|_{H^{2-\delta}(\Omega_i)}.$$

Interpolating estimates (5.6) and (5.7) gives the desired result (5.3). This completes the proof of the theorem. □

In the rest of this section, we give a proof of Proposition 5.1. This follows immediately from the three lemmas below. The first lemma follows easily from the extension noted at the beginning of this section.

Lemma 5.1 *Given $\sigma_{ij} \in H^{1/2}_{00}(\gamma_{ij})$ for $\gamma_{ij} \subset \Gamma$, there exists a $v \in V$ such that*

$$[v]_{\gamma_{ij}} = \sigma_{ij}$$

and

$$\|v\|_V \preceq \left(\sum \|\sigma_{ij}\|^2_{H^{1/2}_{00}(\gamma_{ij})}\right)^{1/2}.$$

Next, for $\mu \in L^2(\gamma_{ij})$ we define the norm

$$\|\mu\|_{Q_{ij}} := \left(\|\mu\|^2_{H^{-1/2}(\gamma_{ij})} + \epsilon\|\mu\|^2_{0,\gamma_{ij}}\right)^{1/2}$$

and its dual

$$\|\mu\|_{Q^*_{ij}} := \sup_{\lambda \in L^2(\gamma_{ij})} \frac{(\lambda, \mu)_{0,\gamma_{ij}}}{\|\lambda\|_{Q_{ij}}}.$$

Recall that the space Q and its dual have been defined in Section 4. We then have the following lemma.

Lemma 5.2 *For all $\lambda \in Q^*$,*

$$\|\lambda\|_{Q^*} \preceq \left(\sum_{\gamma_{ij}} \|\lambda\|^2_{Q^*_{ij}}\right)^{1/2}. \tag{5.8}$$

Proof. Let $\mu \in Q$ be non-zero. First, we verify that

$$\sum_{\gamma_{ij}} \sup_{\sigma_{ij} \in H_{00}^{1/2}(\gamma_{ij})} \frac{(\mu, \sigma_{ij})_{0,\gamma_{ij}}^2}{\|\sigma_{ij}\|^2_{H_{00}^{1/2}(\gamma_{ij})}} \preceq \sup_{v \in V} \frac{(\mu, [v])_{0,\Gamma}^2}{\|v\|_V^2} \tag{5.9}$$

Set $\bar{\sigma}_{ij} = \alpha \sigma_{ij}$, where α is chosen such that $\|\bar{\sigma}_{ij}\|^2_{H_{00}^{1/2}(\gamma_{ij})} = (\mu, \bar{\sigma}_{ij})_{0,\gamma_{ij}}$. By Lemma 5.1, there exists an extension $v \in V$ such that

$$[v]_{\gamma_{ij}} = \bar{\sigma}_{ij} \qquad \text{and} \qquad \|v\|_V^2 \preceq \sum_{\bar{\gamma}_{ij}} \|\bar{\sigma}_{ij}\|^2_{H_{00}^{1/2}(\gamma_{ij})}.$$

Then,

$$\begin{aligned}
&\sum_{\gamma_{ij}} \frac{(\mu, \sigma_{ij})_{0,\gamma_{ij}}^2}{\|\sigma_{ij}\|^2_{H_{00}^{1/2}(\gamma_{ij})}} \\
&= \sum_{\gamma_{ij}} (\mu, \bar{\sigma}_{ij})_{0,\gamma_{ij}} = (\mu, [v])_{0,\Gamma} = \frac{(\mu, [v])_{0,\Gamma}^2}{\|v\|_V^2} \, \frac{\|v\|_V^2}{(\mu, [v])_{0,\Gamma}} \\
&\preceq \frac{(\mu, [v])_{0,\Gamma}^2}{\|v\|_V^2} \, \frac{\sum_{\gamma_{ij}} \|\bar{\sigma}_{ij}\|^2_{H_{00}^{1/2}(\gamma_{ij})}}{\sum_{\gamma_{ij}} (\mu, \bar{\sigma}_{ij})_{0,\gamma_{ij}}} = \frac{(\mu, [v])_{0,\Gamma}^2}{\|v\|_V^2}.
\end{aligned}$$

The inequality (5.9) follows.

It immediately follows from (5.9) that

$$\sum_{\gamma_{ij}} \|\mu\|_{Q_{ij}}^2 \preceq \|\mu\|_Q^2. \tag{5.10}$$

We continue with

$$\begin{aligned}
(\lambda, \mu)_{0,\Gamma} &= \sum_{\gamma_{ij}} (\lambda, \mu)_{0,\gamma_{ij}} \le \sum_{\gamma_{ij}} \|\lambda\|_{Q_{ij}^*} \|\mu\|_{Q_{ij}} \\
&\le \Big(\sum_{\gamma_{ij}} \|\lambda\|^2_{Q_{ij}^*} \Big)^{1/2} \Big(\sum_{\gamma_{ij}} \|\mu\|^2_{Q_{ij}} \Big)^{1/2} \\
&\preceq \Big(\sum_{\gamma_{ij}} \|\lambda\|^2_{Q_{ij}^*} \Big)^{1/2} \|\mu\|_Q.
\end{aligned}$$

The lemma follows dividing by $\|\mu\|_Q$ and taking the supremum. □

Lemma 5.3 *For* $\lambda \in L^2(\gamma_{ij})$*,*

$$\|\lambda\|_{Q_{ij}^*} \preceq \log \epsilon^{-1} \, \|\lambda\|_{H^{1/2}(\gamma_{ij})}. \tag{5.11}$$

Proof. The proof of this lemma is based on techniques from the analysis of domain decomposition preconditioners. We illustrate the proof in the case of three spatial dimensions. The two dimensional case is similar.

Let λ be in $L^2(\gamma_{ij})$ and S_ϵ be a finite element subspace of $H^1(\gamma_{ij})$ of quasi-uniform mesh-size ϵ. The L^2-orthogonal projection operator $\mathcal{Q}$ onto S_ϵ is bounded on $H^{1/2}(\gamma_{ij})$ and satisfies

$$\epsilon^{-1/2}\|\lambda - \mathcal{Q}\lambda\|_{0,\gamma_{ij}} + \|\mathcal{Q}\lambda\|_{H^{1/2}(\gamma_{ij})} \le c\|\lambda\|_{H^{1/2}(\gamma_{ij})}. \tag{5.12}$$

We first split $\lambda = (\lambda - \mathcal{Q}\lambda) + \mathcal{Q}\lambda$, and further decompose the finite element part

$$\mathcal{Q}\lambda = \lambda_1 + \lambda_2$$

such that $\lambda_1 = \mathcal{Q}\lambda$ on $\partial\gamma_{ij}$ and $\lambda_1 = 0$ on all interior nodes of γ_{ij} (λ_2 being the remainder vanishing at $\partial\gamma_{ij}$).

A simple transformation argument and Lemma 4.2 of [8] gives

$$\|\lambda_1\|_{0,\gamma_{ij}} \preceq \epsilon^{1/2}\|\lambda_1\|_{L^2(\partial\gamma_{ij})} \preceq \epsilon^{1/2}\,(\log\epsilon^{-1})^{1/2}\,\|\mathcal{Q}\lambda\|_{H^{1/2}(\gamma_{ij})}.$$

Lemma 4.3 of [8] gives

$$\|\lambda_2\|_{H^{1/2}_{00}(\gamma_{ij})} \preceq \log\epsilon^{-1}\,\|\mathcal{Q}\lambda\|_{H^{1/2}(\gamma_{ij})}.$$

Now, we use the above splitting to get

$$\begin{aligned}
\|\lambda\|_{Q^*_{ij}} &= \sup_{\mu\in L^2(\gamma_{ij})} \frac{(\lambda,\mu)_{0,\gamma_{ij}}}{\|\mu\|_{Q_{ij}}}\\
&= \sup_{\mu\in L^2(\gamma_{ij})} \frac{(\lambda - \mathcal{Q}\lambda,\mu)_{0,\gamma_{ij}} + (\lambda_1,\mu)_{0,\gamma_{ij}} + (\lambda_2,\mu)_{0,\gamma_{ij}}}{\|\mu\|_{H^{-1/2}(\gamma_{ij})} + \epsilon^{1/2}\|\mu\|_{0,\gamma_{ij}}}.
\end{aligned}$$

Further, using the estimate (5.12) we have

$$\begin{aligned}
&\sup_{\mu\in L^2(\gamma_{ij})} \frac{(\lambda - \mathcal{Q}\lambda,\mu)_{0,\gamma_{ij}}}{\|\mu\|_{H^{-1/2}(\gamma_{ij})} + \epsilon^{1/2}\|\mu\|_{0,\gamma_{ij}}}\\
&\quad\le \sup_{\mu\in L^2(\gamma_{ij})} \frac{\|\lambda - \mathcal{Q}\lambda\|_{0,\gamma_{ij}}\|\mu\|_{0,\gamma_{ij}}}{\|\mu\|_{H^{-1/2}(\gamma_{ij})} + \epsilon^{1/2}\|\mu\|_{0,\gamma_{ij}}}\\
&\quad\preceq \|\lambda\|_{H^{1/2}(\gamma_{ij})}.
\end{aligned}$$

Similarly, using the estimates for λ_1 and λ_2 we get

$$(\lambda_1,\mu)_{0,\gamma_{ij}} \le \|\lambda_1\|_{0,\gamma_{ij}}\|\mu\|_{0,\gamma_{ij}} \preceq \epsilon^{1/2}\,(\log\epsilon^{-1})^{1/2}\,\|\lambda\|_{H^{1/2}(\gamma_{ij})}\|\mu\|_{0,\gamma_{ij}}$$

and

$$\begin{aligned}
(\lambda_2,\mu)_{0,\gamma_{ij}} &\le \|\lambda_2\|_{H^{1/2}_{00}(\gamma_{ij})}\|\mu\|_{H^{-1/2}(\gamma_{ij})}\\
&\preceq \log\epsilon^{-1}\,\|\mathcal{Q}\lambda\|_{H^{1/2}(\gamma_{ij})}\|\mu\|_{H^{-1/2}(\gamma_{ij})}.
\end{aligned}$$

Finally, combining the estimates for all three parts, we complete the proof:

$$\|\lambda\|_{Q^*_{ij}} = \sup_{\mu \in L^2(\gamma_{ij})} \frac{(\lambda, \mu)_{0,\gamma_{ij}}}{\|\mu\|_{Q_{ij}}} \preceq \log \epsilon^{-1} \|\lambda\|_{H^{1/2}(\gamma_{ij})}.$$

□

In the case without cross-points, we can get a slightly better result. In this case, $\gamma_{ij} = \Gamma$. The following theorem provides an error estimate in this case.

Theorem 5.2 *In the case of absence of "cross-points," the following estimate holds*

$$||e||_V + ||E||_Q \preceq \epsilon^{1-\delta} ||u||_{H^{2-\delta}(\Omega)} \tag{5.13}$$

for $u \in H^{2-\delta}(\Omega)$, $0 \leq \delta < 1/2$.

Proof. Because there are no "cross-points" for $v \in V$, the jump $[v] = \Lambda v$ is in $H^{1/2}_{00}(\Gamma)$. Therefore, there is an extension, which satisfies (5.1) so that

$$||q||_Q \succeq \sup_{v \in V} \frac{c(\Lambda v, q)}{||v||_V} \succeq ||q||_{H^{-1/2}(\Gamma)}.$$

This implies

$$\sup_{q \in Q} \frac{c(\theta, q)}{||q||_Q} \preceq \sup_{q \in Q} \frac{c(\theta, q)}{||q||_{H^{-1/2}(\Gamma)}} \preceq ||\theta||_{H^{1/2}_{00}(\Gamma)}.$$

so that

$$||e||_V + ||E||_Q \preceq \epsilon ||\theta||_{H^{1/2}_{00}(\Gamma)}. \tag{5.14}$$

Interpolating (5.6) and (5.14) we get

$$||e||_V + ||E||_Q \preceq \epsilon^{1-\delta} ||\theta||_{H^{1/2-\delta}(\Gamma)}.$$

The result then follows from the trace estimate

$$\|\theta\|_{H^{1/2-\delta}(\Gamma)} \preceq \|u\|_{H^{2-\delta}(\Omega)},$$

which holds for polygonal interface Γ (cf. [17]). □

6 Finite element approximation of the penalty formulation

6.1 Finite element formulation and error analysis

Now, we discretize the problem (3.1) by the finite element method. Each subdomain Ω_i is meshed independently by a quasi-uniform and shape-regular triangulation $\mathcal{T}_i$, and consequently, the whole domain has a finite element splitting $\mathcal{T} = \cup_i \mathcal{T}_i$. Quasi-uniformity of the mesh means that for $\tau \in \mathcal{T}$ and $h_\tau = diam(\tau)$, $|\tau| = meas(\tau)$ we have $|\tau| \approx h_\tau^d$, where $d = 2, 3$ is the dimension of the space. We shall use also the global mesh-size parameter

$$h = \max_{\tau \in \mathcal{T}} h_\tau.$$

Our analysis uses the condition that the mesh $\mathcal{T}$ is globally quasi-uniform, i.e., $h \approx h_\tau$ for all $\tau \in \mathcal{T}$. We stress again, there is no assumption that along an interface γ_{ij}, the triangulations $\mathcal{T}_i$ and $\mathcal{T}_j$ produce the same mesh.

Let $V_{i,h}$ be the conforming (see [13]) finite element space of piecewise linear functions associated with the triangulation $\mathcal{T}_i$. Further, let $V_h : V_h|_{\Omega_i} = V_{i,h}$, for $i = 1, \ldots, p$, be the finite element space on $\mathcal{T}$. The functions in V_h are, in general, discontinuous across γ_{ij}. However, their traces on γ_{ij} from Ω_i and Ω_j are well-defined.

Let $I_h : V \to V_h$ be an operator such that for $u \in H^{2-\beta}(\Omega)$ and $0 \le \beta \le 1$:

$$\begin{aligned} & h^{-1} \|u - I_h u\|_{L^2(\Omega)} + \|u - I_h u\|_{H^1(\Omega)} + h^{-1/2} \|u - I_h u\|_{L^2(\Gamma)} \\ & \preceq h^{1-\beta} \|u\|_{H^{2-\beta}(\Omega)}. \end{aligned} \tag{6.1}$$

Now, the interior penalty finite element method reads as: Find $u_h^\epsilon \in V_h$ such that

$$A^\epsilon(u_h^\epsilon, \phi) := a(u_h^\epsilon, \phi) + \epsilon^{-1} c(\Lambda u_h^\epsilon, \Lambda \phi) = f(\phi) \quad \text{for all } \phi \in V_h. \tag{6.2}$$

Obviously, the bilinear form $A^\epsilon(\cdot, \cdot)$ is symmetric and positive definite on $V_h \times V_h$. Therefore, the corresponding finite element “stiffness” matrix is symmetric and positive definite, and the finite element system has a unique solution.

Now, we derive an error estimate for the finite element interior penalty method. According to our construction $V = \sum H^1(\Omega_i) \cap H_0^1(\Omega)$ and

$$(w, v)_V = \sum_i \int_{\Omega_i} (\nabla w \cdot \nabla v + wv)\, dx.$$

Because the number of subdomains p is finite and all Ω_i are shape-regular, it follows that $A^1(v, v)$ is uniformly equivalent to the norm $\|v\|_V^2$, and the inequality (4.1) holds. Therefore, the results of the previous sections are valid, and we can apply Theorem 5.1.

The error estimate is almost an immediate consequence of Theorem 5.1 and the approximation property (6.1) of the space V_h. Indeed, the error $u_\epsilon - u_h^\epsilon$ satisfies the orthogonality property

$$A_h^\epsilon(u_\epsilon - u_h^\epsilon, \phi) = 0 \text{ for all } \phi \in V_h.$$

Using the coercivity of $A^\epsilon(\cdot,\cdot)$, we get

$$\begin{aligned}\|u_\epsilon - u_h^\epsilon\|_V^2 &\preceq A^\epsilon(u_\epsilon - u_h^\epsilon, u_\epsilon - u_h^\epsilon)\\ &\preceq \inf_{v\in V_h} A^\epsilon(u_\epsilon - v, u_\epsilon - v)\\ &\preceq A^\epsilon(u_\epsilon - u, u_\epsilon - u) + A^\epsilon(u - I_h u, u - I_h u).\end{aligned}$$

Now the estimates (5.3) and (6.1) produce the following result:

$$\begin{aligned}\|u - u_h^\epsilon\|_V &\preceq \|u - u_\epsilon\|_V + \|u_\epsilon - u_h^\epsilon\|_V\\ &\preceq (\epsilon^{1-\delta}|\log\epsilon|^{1-2\delta} + h^{1-\delta} + \epsilon^{-1/2}h^{3/2-\delta})\|u\|_{H^{2-\delta}(\Omega)}\end{aligned}$$

for $u \in H^{2-\delta}(\Omega)$, $0 \le \delta < 1/2$.

The above estimates suggest that for the penalty parameter $\epsilon \approx h$ we get an almost optimal convergence rate. Thus, we shall use $\epsilon = \beta^{-1}h$ with β, a real number. The following theorem is a corollary of the above estimate:

Theorem 6.1 *Assume that the solution u of the problem (2.1) belongs to $H^{2-\delta}(\Omega)$ for some $0 \le \delta < 1/2$. Then the solution $u_h \in V_h$ of the interior penalty finite element method*

$$a_h(u_h, \phi) + \beta h^{-1} c(\Lambda u_h, \Lambda\phi) = f(\phi) \quad \textit{for all } \phi \in V_h$$

exists and satisfies the a priori error estimate

$$\|u - u_h\|_V \preceq h^{1-\delta}|\log h|^{1-2\delta}\|u\|_{H^{2-\delta}(\Omega)}.$$

Moreover, the condition number of the corresponding finite element "stiffness" matrix is the same as in the case of standard Galerkin method with linear elements, namely, $O(h^{-2})$.

6.2 Numerical tests

In this subsection, the performance of the proposed penalty method is reported on two model examples for the Poisson equation on the unit square with Dirichlet boundary conditions. Our finite element implementation handles arbitrary triangulations of the domain and linear finite elements.

In the table below, we present the error $u - u_h$ measured in discrete L^2 and H^1-norms for two test problems for the Poisson equation. For both examples we have set $\beta = 1$. The domain is split into four equal subdomains that are triangulated independently so that the meshes do not match along the interface Γ. The test problems are designed to check the accuracy of the interior penalty method. The first example has exact solution $u(x_1, x_2) = \sin^2(2\pi x_1)\sin^2(2\pi x_2)$ so that the normal derivative along the interfaces γ_{ij} is zero. This means that the interior penalty method should have the same accuracy as the standard Galerkin method in both L^2- and H^1-norms. This is readily observed in Table 1. The second test problem has exact solution $u(x_1, x_2) = x_1^2 + x_2^2$. We have observed from our computations that the interface is the main contributor to the error.

Table 1. Numerical results for four subdomains with non-matching grids

		exact solution u $\sin^2(2\pi x_1)\sin^2(2\pi x_2)$			exact solution u $x_1^2 + x_2^2$			
level	# nodes	L^2-error	H^1-error	ratio	L^2-error	H^1-error	ratio	cond. #
1	96	0.05536	1.09488		0.00471	0.06638		30
2	225	0.01567	0.58970	1.85	0.00295	0.03764	1.76	105
3	833	0.00401	0.30059	1.96	0.00105	0.02085	1.80	439
4	3201	0.00101	0.15108	1.99	0.00054	0.01145	1.81	1829
5	12545	0.00025	0.07564	2.00	0.00028	0.00639	1.80	7385
6	52695	0.00006	0.03783	2.00	0.00014	0.00353	1.81	29438
order		$\approx$ 2	$\approx$ 1		$\approx$ 1	$\approx$ 0.91		

Note that the convergence in L^2-norm is of first order, while the convergence in H^1-norm is approximately first order. In the discrete L^2- and H^1-norms, the relative error on the finest (6th) level is 0.03% and 1.95% for the exact solution $u(x_1, x_2) = \sin^2(2\pi x_1)\sin^2(2\pi x_2)$ and 0.08% and 0.72% for the exact solution $u(x_1, x_2) = x_1^2 + x_2^2$.

In addition, we ran numerical experiments which involved changing the weight β in the penalty term. The number of nodes were kept fixed at about 820 and 830, correspondingly, for matching and non-matching grids, which gives rise to $h \approx 0.04$. In both cases the exact solution was $u(x_1, x_2) = x_1^2 - x_2^2$. The computations, presented in Tables 2 and 3, show that by increasing β we put more weight on the penalty term, and as expected, this leads to decreasing the error in all norms. However, for β larger than $h^{-1/2}$, there is no significant improvement in the accuracy. Moreover, in the case of non-matching grids very large β causes deterioration of the error in maximum and H^1-norm, and as expected, increases the condition number.

More examples and computational results, including condition number and errors in various norms, are reported in [21].

Table 2. Varying β for matching grids with 820 points

β	L^∞-error	L^2-error	H^1-error	condition #
0.1	0.26508	0.06434	0.36322	440
1	0.03589	0.00915	0.07158	440
$h^{-1/2} \approx 5$	0.00798	0.00191	0.03802	615
10	0.00411	0.00101	0.03574	1162
1000	0.00004	0.00027	0.03485	42261

Table 3. Varying β for non-matching grids with 833 points

β	L^∞-error	L^2-error	H^1-error	condition #
0.1	0.86032	0.36833	1.14018	420
1	0.13809	0.04873	0.26776	420
$h^{-1/2} \approx 5$	0.03429	0.01038	0.16705	1190
10	0.01859	0.00586	0.15974	2273
1000	0.02523	0.00486	0.16925	88288

Acknowledgements. Part of this research has been done during the summer visits of the first and the second authors to Lawrence Livermore National Laboratory. The authors thank the Institute for Scientific Computing Research and the Center for Applied Scientific Computing for their hospitality and for the technical and financial support.

References

[1] Arbogast, T., Cowsar, L.C., Wheeler, M.F., Yotov, I.: Mixed finite element methods on non-matching multi-block grids. SIAM J. Numer. Anal. **37**, 1295–1315 (2000)
[2] Arnold, D.: An interior penalty finite element method with discontinuous elements. SIAM J. Numer. Anal. **19**, 742–760 (1982)
[3] Arnold, D., Brezzi, F., Cockburn, B., Marini, L.D.: Unified analysis of discontinuous Galerkin methods for elliptic problems. SIAM J. Numer. Anal. **39**(5), 1749–1779 (2001)
[4] Babuška, I.: The finite element method for elliptic equations with discontinuous coefficients. Comput. **5**, 207–213 (1970)
[5] Babuška, I.: The finite element method with penalty. Math. Comp. **27**(122), 221–228 (1973)
[6] Bjorstad, P.E., Espedal, M.S., Keyes, D.E.: 9th International Conference on Domain Decomposition Methods. Norway: Ullensvang, 1996, Published by ddm.org, 1998
[7] Bramble, J.H., Pasciak, J.E., Schatz, A.: The construction of preconditioners for elliptic problems by substructuring, I. Math. Comp. **47**, 103–134 (1986)
[8] Bramble, J.H., Pasciak, J.E., Schatz, A.: The construction of preconditioners for elliptic problems by substructuring, IV. Math. Comp. **53**, 1–24 (1989)
[9] Bramble, J.H., Pasciak, J.E., Vassilevski, P.S.: Computational scales of Sobolev norms with application to preconditioning. Math. Comp. **69**, 463–480 (2000)

[10] Brezzi, F., Fortin, M.: Mixed and Hybrid Finite Element Methods. Berlin-Heidelberg-New York: Springer, 1991
[11] Cao, Y., Gunzburger, M.D.: Least–squares finite element approximations to solutions of interface problems. SIAM J. Numer. Anal. **35**, 393–405 (1998)
[12] Chan, T.F., Kako, T., Kawarada, H., Pironneau, O.: Domain Decomposition Methods in Science and Engineering. 12-th International Conference in Chiba, Japan, 1999, Published by ddm.org, 2001
[13] Ciarlet, P.G.: The finite element method for elliptic problems. Amsterdam: North Holland, 1978
[14] Cockburn, B., Dawson, C.: Approximation of the velocity by coupling discontinuous Galerkin and mixed finite element methods for flow problems. Preprint, 2001
[15] Douglas, J., Dupont, T.: Interior penalty procedures for elliptic and parabolic Galerkin methods. Lecture Notes in Physics **58**, 207–216 (1978)
[16] Feistauer, M., Felcman, J., Lucacova-Medvidova, M., Warenicke, G.: Error estimates of a combined finite volume - finite element method for nonlinear convection-diffusion problems. Preprint, 2001
[17] Grisvard, P.: Elliptic Problems in Non-smooth Domains. Boston: Pitman, 1985
[18] Lai, C.-H., Bjorstad, P., Cross, M., Widlund, O.: Eleventh Int. Conference on Domain Decomposition Methods. UK: Greenwich, 1998, Published by ddm.org, 1999
[19] Lazarov, R.D., Pasciak, J.E., Vassilevski, P.S.: Iterative solution of a coupled mixed and standard Galerkin discretization method for elliptic problems. Numer. Lin. Alg. Appl. **8**, 13–31 (2001)
[20] Lazarov, R.D., Pasciak, J.E., Vassilevski, P.S.: Mixed finite element methods for elliptic problems on non-matching grids. In: Large-Scale Scientific Computations of Engineering and Environmental Problems II M. Griebel, et al., (Eds), Vieweg, Notes on Numerical Fluid Mechanics **73**, 2000, pp. 25–35
[21] Lazarov, R.D., Tomov, S.Z., Vassilevski, P.S.: Interior penalty discontinuous approximations of elliptic problems. Comput. Method Appl. Math. **1**(4), 367–382 (2001)
[22] Lions, J.-L.: Problèmes aux limites nonhomogène à donées irrégulières; Une méthode d'approximation. In: Numerical Analysis of Partial Differential Equations C.I.M.E. 2 Ciclo, Ispra, 1967 (ed), Edizioni Cremonese, Rome, 1968, pp. 283–292
[23] Lions, J.-L., Peetre, J.: Sur une class d'espaces d'interpolation, Institute des Hautes Etudes Scientifique. Publ. Math. **19**, 5–68 (1994)
[24] Nitsche, J.: Uber ein Variationsprinzip zur Losung von Dirichlet-Problemen bei Verwendung von Teilraumen, die keinen Randbedingungen unterworfen sind. Abh. Math. Sem. Univ. Hamburg **36** (Collection of articles dedicated to Lothar Collatz on his sixtieth birthday) 1971, 9–15
[25] Rivière, B., Wheeler, M.F., Girault, V.: Improved energy estimates for the interior penalty, constrained and discontinuous Galerkin methods for elliptic problems. Part I, Computational Geosciences **3**(3,4), 337–360 (1999)
[26] Rusten, T., Vassilevski, P.S., Winther, R.: Interior penalty preconditioners for mixed finite element approximations of elliptic problems. Math. Comp. **65**, 447–466 (1996)
[27] Schöberl, J.: Robust Multigrid Methods for Parameter Dependent Problems. Ph. D. Thesis, Linz: University of Linz, June, 1999
[28] Wieners, C., Wohlmuth, B.I.: The coupling of mixed and conforming finite element discretizations. In: Domain Decomposition Methods 10, J. Mandel, C. Farhat, X.-C. Cai, (eds) Contemporary Math. **218**, 1998, pp. 547–554